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HEAT TRANSFER THROUGH DROP CONDENSATE USING DIFFERENTIAL INEQUALITIES

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Abstract—The quasisteady Nusselt number for a drop of condensate is calculated analytically for an arbitrary contact angle in the range $[0, \pi/2]$ using a spherical segment geometry. Uniform base temperature is assumed and to ensure the boundedness of the total heat flux the convection boundary condition is used at the free surface. Differential inequalities are used to obtain error bounds which approach zero as the contact angle tends to $\pi/2$. For small contact angles the calculated Nusselt number tends to the exact value for a flat-disk droplet.

NOMENCLATURE

Bi, Biot number;
D, spatial domain defined by the interior of the droplet;
∂D_1, boundary defined by the flat base of the droplet;
∂D_2, boundary defined by the free surface of the droplet;
h, heat-transfer coefficient at the liquid–vapor interface (free surface);
k, thermal conductivity of the droplet liquid;
Nu, Nusselt number;
ΔNu, difference between the bounds on Nu;
P_n, Legendre polynomial of the first kind, nth order;
Q, total heat flow through the droplet;
$\bar{Q}$, upper bound on Q;
$\underline{Q}$, lower bound on Q;
q, local heat flux across the base;
$\bar{q}$, upper bound on q;
$\underline{q}$, lower bound on q;
R, base radius of the droplet;
R', radius of curvature of droplet;
r, dimensionless radial coordinate;
$\mathbf{r}$, dimensionless position vector;
T, temperature distribution in the droplet;
$\bar{T}$, upper bound on T;
$\underline{T}$, lower bound on T;
T_a, ambient temperature;
T_b, base temperature distribution;
T_{b0}, constant base temperature;
T_{exact}, exact temperature distribution for the proposed model;
$\bar{T}_b$, upper bound on T_b;
$\underline{T}_b$, lower bound on T_b;
ΔT, difference between the vapor and the base temperature;
t, time.

Greek symbols

θ, angular coordinate;
θ_0, contact angle;
λ, latent heat of vaporization of droplet liquid;
μ, $\cos\theta$;
μ_0, $\cos\theta_0$;
ρ, density of droplet liquid.

1. INTRODUCTION

THE SIGNIFICANTLY higher heat flux observed for dropwise (as opposed to filmwise) condensation has inspired considerable research effort to better understand the fundamental aspects of droplet formation and growth. Although the process is exceedingly complex, some simplification of the analyses is possible using the results of Graham and Griffith [1] who showed that most of the heat is transferred through droplets of diameter less than 100 μm. For such small droplets the influence of gravity on the droplet shape is negligible and a spherical-segment geometry may be assumed. If it is further assumed that the heat transfer and droplet growth are quasisteady the process is governed by the steady heat-conduction equation and an analytic formulation can be completed with various boundary conditions.

Fatica and Katz [2] and others [3, 4] used constant-temperature boundary conditions on the droplet base and free surface for arbitrary contact angle. However, the discontinuity in the boundary temperature along the edge of the base prevents the computation of a finite value of the heat flux (see Appendix B). Valid solutions for the heat flux (required to compute the droplet growth) can be obtained by requiring that the boundary temperature be continuous. This condition can be achieved by assuming a convective boundary condition with a finite heat-transfer coefficient on the

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free surface. Umur and Griffith [5] used this boundary condition together with uniform base temperature to obtain an exact solution for a hemispherical droplet which corresponds to a contact angle of 90°. Hemispherical droplets were also analyzed by Hurst and Olson [6] using the finite element method and they overcame the temperature discontinuity problem by considering heat conduction through the condenser plate as well as the droplet. However, for different contact angles the only valid results were obtained numerically by Ahrendts [7].

In the present analysis the theory of differential inequalities [8, 9] is applied to obtain approximate solutions which are upper and lower bounds for the exact solutions of the temperature distributions in droplets of arbitrary contact angle $0 \leqslant \theta_0 \leqslant \pi/2$. Here the exact solution is defined as the one corresponding to zero base temperature and a convective boundary condition at the free surface. It is further shown that these approximate solutions become exact for $\theta_0 = 0$ and $\theta_0 = \pi/2$. Upper and lower bounds on the droplet heat flux are also derived and finally an approximate expression for the droplet Nusselt number as a function of the Biot number and contact angle is presented with rigorous error bounds.

2. MATHEMATICAL FORMULATION

The growth of a single droplet depends on the overall heat flux and, hence, it is necessary to find the droplet temperature distribution $T^*(\mathbf{r}^*)$. As a reasonable physical model it is assumed that the base of the droplet is at constant temperature T_{b0} and that the free surface is exposed to constant ambient temperature T_a. Under the quasisteady assumption the dimensionless temperature

$$T(\mathbf{r}) = (T^* - T_{b0})/\Delta T \text{ (where } \Delta T = T_a - T_{b0})$$

at any instant of the development of a droplet of radius R' is the solution to

$$\nabla^2 T(\mathbf{r}) = 0 \quad \text{in } D \tag{1}$$

$$T(\mathbf{r})|_{\partial D_1} = 0 \tag{2}$$

$$-\frac{\partial T(\mathbf{r})}{\partial n}\Big|_{\partial D_2} = \frac{h}{kR'}(T|_{\partial D_2} - 1) \tag{3}$$

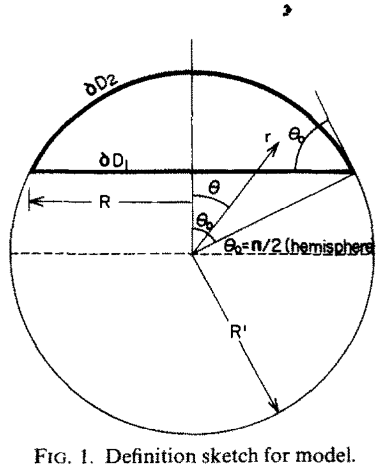

FIG. 1. Definition sketch for model.

where D is the interior of the droplet, ∂D_1 is the flat base of the droplet, ∂D_2 is the free surface of the droplet, k is the thermal conductivity of the droplet liquid, h is the heat-transfer coefficient at the free surface, $T(\mathbf{r})$ is the dimensionless temperature distribution, $\partial T(\mathbf{r})/\partial n$ is the outward normal derivative, and $\mathbf{r} = \mathbf{r}^*/R'$ is the dimensionless position vector.

If spherical coordinates are used for this problem, the boundary ∂D_1 cannot be defined by a single coordinate except in the special case of a hemisphere (see Fig. 1). Consequently, the boundary condition (2) cannot be exactly satisfied except when the droplet is a hemisphere. However, it can be approximately satisfied in this coordinate system if a sphere in a discontinuous ambient temperature is considered as follows

$$\nabla^2 T(r, \mu) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial T}{\partial \mu}\right] = 0, \tag{4}$$

$$\frac{\partial T(r, \mu)}{\partial r}\Big|_{r=1} = \frac{h}{kR'}[T(1, \mu) - T_a(\mu)], \tag{5}$$

where $T_a(\mu)$ is the ambient temperature and $\mu = \cos\theta$. In the region ∂D_2 the equations (3) and (5) must be identical and therefore, if θ_0 is the contact angle, then

$$T_a(\mu) = 1 \text{ for } \mu_0 < \mu \leqslant 1, \tag{6}$$

where $\mu_0 = \cos\theta_0$.

Over the rest of the surface of the sphere $T_a(\mu)$ must be chosen so that (2) is satisfied, at least approximately. Even if nonzero constant base temperature is found the condition (2) can be satisfied by subtracting that constant and scaling the resulting function. Therefore, the problem now is to find $T_a(\mu), -1 \leqslant \mu < \mu_0$ such that the base temperature is as uniform as possible. If as a first attempt it is assumed that

$$T_a(\mu) = 0 \quad -1 \leqslant \mu < \mu_0, \tag{7}$$

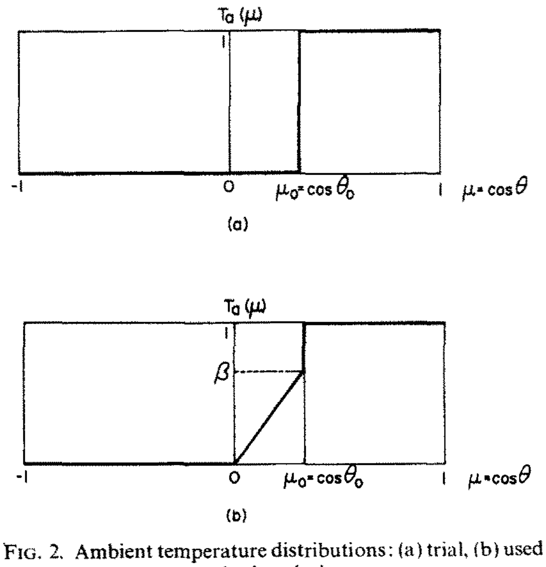

FIG. 2. Ambient temperature distributions: (a) trial, (b) used to obtain solutions.

(see Fig. 2a) then for large but finite values of h the temperature at the edge of the droplet $T(1, \mu_0)$ takes on approximately the average value of $T_a(\mu)$ at $\mu = \mu_0$ for any μ_0, i.e.

$$T(1, \mu_0) \simeq \tfrac{1}{2}. \tag{8}$$

The temperature at the centre of the base is also close to 1/2 for $\mu_0 \sim 0$, i.e. contact angle $\theta_0 \sim \pi/2$. However, as θ_0 approaches 0, this temperature approaches unity and since the edge temperature is close to 1/2 for all contact angles, the uniformity of the base temperature becomes quite poor. Thus to assure uniformity of the base temperature, the edge temperature must increase with decreasing θ_0. This may be achieved by weakening the discontinuity in $T_a(\mu)$ at $\mu = \mu_0$ by increasing $T_a(\mu)$ linearly with μ from $T_a(0) = 0$ to a suitable constant $T_a(\mu_0) = \beta(\mu_0)$ depending on μ_0 as μ increases from 0 to μ_0, i.e.

$$T_a(\mu) = \begin{cases} 1 & \mu_0 < \mu \leqslant 1, \\ \beta(\mu_0)\mu/\mu_0 & 0 < \mu < \mu_0, \\ 0 & -1 \leqslant \mu < 0, \end{cases} \tag{9}$$

(see Fig. 2b).

For $\theta_0 = \pi/2$ (i.e. $\mu_0 = 0$) equation (7) leads to uniform base temperature and no correction for $T_a(\mu)$ is needed. Therefore, in order that (7) and (9) be identical for $\mu_0 = 0$, it is necessary that $\beta(0) = 0$. Since the temperature at the centre of the base approaches unity as $\theta_0 \to 0$, good uniformity of the base temperature can be achieved by requiring that the edge temperature also approaches unity for $\theta_0 \to 0$ and this makes it necessary that $\beta(1) = 1$. Under these conditions, a suitable expression for $\beta(\mu_0)$ is found to be

$$\beta(\mu_0) = \left| 1 - \frac{2\cos^{-1}\mu_0}{\pi} \right|^{2\cos^{-1}\mu_0},$$

i.e.

$$\beta(\theta_0) = \left| 1 - \frac{2\theta_0}{\pi} \right|^{2\theta_0/\pi}. \tag{10}$$

Using the expression (9) for $T_a(\mu)$ in equation (5) the following expression for $T(r, \mu)$ is found

$$T(r, \mu) = a_0 + \sum_{n=1}^{\infty} a_n r^n P_n(\mu) \tag{11}$$

where

$$a_0 = \tfrac{1}{2}(1 - \mu_0) + \tfrac{1}{4}\beta\mu_0^2, \tag{12}$$

$$a_1 = \frac{\tfrac{3}{4}(1 - \mu_0^2) + \tfrac{1}{2}\beta\mu_0^2}{1 + (1 - \mu_0^2)^{1/2}/Bi}, \tag{13}$$

and

$$a_n = \frac{(2n+1)}{2} \frac{\left| (1 - \mu_0^2)\dfrac{P_n'(\mu_0)}{n(n+1)} + \dfrac{\beta(\mu_0)}{\mu_0} \cdot \dfrac{(1 - \mu_0^2)[P_n(\mu_0) - \mu_0 P_n'(\mu_0)] - P_n(0)}{n(n+1) - 2} \right|}{\left| 1 + \dfrac{n(1 - \mu_0^2)^{1/2}}{Bi} \right|} \tag{14}$$

$n = 2, 3, 4, \ldots$

Here the Biot number is defined in terms of the base radius R as

$$Bi = \frac{hR}{k} = \frac{hR'(1 - \mu_0^2)^{1/2}}{k}. \tag{15}$$

It is now appropriate to consider how well the solution (11) satisfies the boundary condition (2) on the base of the droplet. After substituting the relation defining the base

$$r = \mu_0/\mu \tag{16}$$

into (11) the expression for the base temperature $T_b(\mu)$ is

$$T_b(\mu) = a_0 + \Sigma a_n \left(\frac{\mu_0}{\mu} \right)^n P_n(\mu). \tag{17}$$

which is presented in Fig. 3 for various values of the Biot number and selected contact angles θ_0.

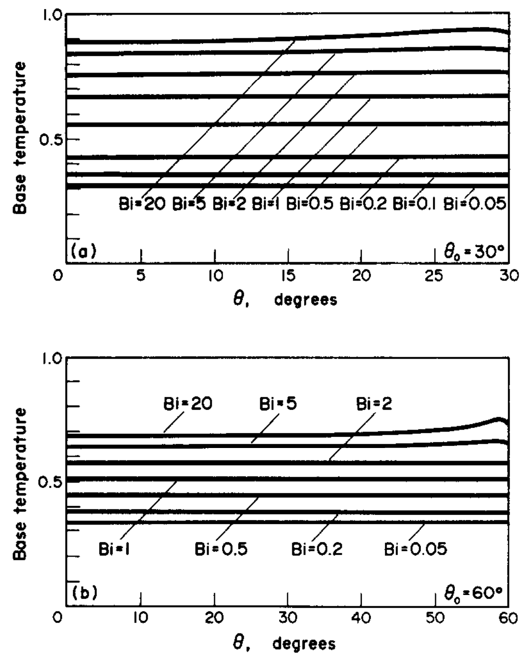

FIG. 3. Base temperature distributions as a function of θ for different Biot numbers: (a) contact angle $\pi/6$, (b) contact angle $\pi/3$.

UPPER AND LOWER BOUNDS

If now for a given value of θ_0 and Bi, the minimum base temperature $\underline{T_b}$ is subtracted from $T(r, \mu)$ and the resulting function divided by a scaling factor $[1 - \underline{T_b}]$, the following function is obtained

$$\bar{T}(r, \mu) = \frac{T(r, \mu) - \underline{T_b}}{1 - \underline{T_b}}. \tag{18}$$

This function exactly satisfies the differential equation

$$\nabla^2 \bar{T}(r,\mu) = 0 \quad \text{in} \quad D \tag{19}$$

and one of the boundary conditions

$$-\frac{\partial \bar{T}(r,\mu)}{\partial r}\Big|_{\partial D_2} = \frac{h}{kR'}[\bar{T}(r,\mu)|_{\partial D_2} - 1], \tag{20}$$

On ∂D_1 (i.e. the base),

$$0 \leqslant \bar{T}(r,\mu)|_{\partial D_1} \tag{21}$$

by the definition of $\underline{T}_b$, i.e. the base boundary condition is satisfied only as an inequality. If the boundary conditions (2) and (3) are written in operator form

$$B[T_{\text{exact}}(r,\mu)] = \begin{cases} \left| \dfrac{\partial T_{\text{exact}}(1,\mu)}{\partial r} + \dfrac{h}{kR'} T_{\text{exact}}(1,\mu) \right| \text{ in } \partial D_2 \\ T_{\text{exact}}(r,\mu)|_{\partial D_1} \end{cases} = \begin{cases} \dfrac{h}{kR'} \text{ in } \partial D_2 \\ 0 \text{ in } \partial D_1 \end{cases} \tag{22}$$

then

$$B[T_{\text{exact}}(r,\mu)] \leqslant B[\bar{T}(r,\mu)] \tag{23}$$

while

$$-\nabla^2 T_{\text{exact}}(r,\mu) = -\nabla^2 \bar{T}(r,\mu) = 0. \tag{24}$$

By applying the theorem stated in Appendix A, it can be shown that as a consequence of (23) and (24),

$$T_{\text{exact}}(r,\mu) \leqslant \bar{T}(r,\mu), \tag{25}$$

i.e. $\bar{T}(r,\mu)$ is the upper bound to the unknown exact solution $T_{\text{exact}}(r,\mu)$ which satisfies (1), (2) and (3).

Similarly, by defining the function

$$\underline{T}(r,\mu) = \frac{T(r,\mu) - \bar{T}_b}{1 - \bar{T}_b} \tag{26}$$

where $\bar{T}_b$ is the maximum base temperature, it can be shown that

$$\nabla^2 \underline{T}(r,\mu) = 0 \text{ in } D, \tag{27}$$

$$-\frac{\partial \underline{T}(r,\mu)}{\partial r}\Big|_{\partial D_2} = \frac{h}{kR'}[\underline{T}(r,\mu)|_{\partial D_2} - 1], \tag{28}$$

and that

$$\underline{T}(r,\mu)|_{\partial D_1} \leqslant 0. \tag{29}$$

Again, application of the theorem, leads to the result that

$$\underline{T}(r,\mu) \leqslant T_{\text{exact}}(r,\mu). \tag{30}$$

From these upper and lower bounds for the temperature, the bounds for the heat flux $q(\mu)$ and the total heat flow Q can be calculated as follows.

Since

$$\underline{T}(r,\mu) \leqslant T_{\text{exact}}(r,\mu) \leqslant \bar{T}(r,\mu), \tag{31}$$

$$\underline{T}(1,\mu) \leqslant T_{\text{exact}}(1,\mu) \leqslant \bar{T}(1,\mu), \tag{32}$$

and

$$h[\underline{T}(1,\mu) - 1] \leqslant h[T_{\text{exact}}(1,\mu) - 1] \leqslant h[\bar{T}(1,\mu) - 1]. \tag{33}$$

By definition

$$q(\mu) = -h[T_{\text{exact}}(1,\mu) - 1]\Delta T \tag{34}$$

and if $\bar{q}(\mu)$ and $\underline{q}(\mu)$ are defined as

$$\bar{q}(\mu) = -h[\underline{T}(1,\mu) - 1]\Delta T \tag{35}$$

and

$$\underline{q}(\mu) = -h[\bar{T}(1,\mu) - 1]\Delta T \tag{36}$$

then

$$\underline{q}(\mu) \leqslant q(\mu) \leqslant \bar{q}(\mu).$$

Consequently, the total heat flow Q has the bounds

$$\underline{Q} = -2\pi R'^2 \int_1^{\mu_0} \underline{q}(\mu)\,d\mu \leqslant Q \leqslant -2\pi R'^2 \int_1^{\mu_0} \bar{q}(\mu)\,d\mu = \bar{Q} \tag{37}$$

where after integration it is found that

$$\underline{Q} = \frac{2\pi R'^2 h}{1 - \underline{T}_b}(1-\mu_0) \times \left| (1-a_0) - (1+\mu_0) \sum_{n=1}^{\infty} \frac{a_n P_n'(\mu_0)}{n(n+1)} \right| \Delta T \tag{38}$$

and

$$\bar{Q} = \frac{2\pi R'^2 h}{1 - \bar{T}_b}(1-\mu_0) \times \left| (1-a_0) - (1+\mu_0) \sum_{n=1}^{\infty} \frac{a_n P_n'(\mu_0)}{n(n+1)} \right| \Delta T. \tag{39}$$

4. DROPLET NUSSELT NUMBER

By defining the Nusselt number as

$$Nu = \frac{Q}{kR\Delta T}, \tag{40}$$

it can be shown that

$$\frac{f(\mu_0)}{1 - \underline{T}_b} \leqslant Nu \leqslant \frac{f(\mu_0)}{1 - \bar{T}_b}, \tag{41}$$

where

$$f(\mu_0) = \frac{2\pi Bi}{(1+\mu_0)} \times \left| (1-a_0) - (1+\mu_0) \sum_{n=1}^{\infty} \frac{a_n P_n'(\mu_0)}{n(n+1)} \right|. \tag{42}$$

The quantities $\underline{T}_b$ and $\bar{T}_b$ appearing in the inequality (41) have to be determined numerically for each value of μ_0 and Bi. Since the base temperature $T_b(\mu)$ is almost uniform (see Fig. 3) both $\bar{T}_b$ and $\underline{T}_b$ can be approximated by the temperature $T_b(\mu_0)$ at the edge for which case

$$\bar{T}_b \simeq \underline{T}_b \simeq T_b(\mu_0) = \sum_{n=0}^{\infty} a_n P_n(\mu_0). \tag{43}$$

The Nusselt number can then be approximated as

$$Nu \simeq \frac{f(\mu_0)}{1-T_h(\mu_0)}$$

i.e.

$$Nu \simeq \frac{2\pi Bi}{(1+\mu_0)} \times \frac{\left|(1-a_0)-(1+\mu_0)\sum_{n=1}^{\infty}\frac{a_n P_n'(\mu_0)}{n(n+1)}\right|}{\left|(1-a_0)-\sum_{n=1}^{\infty} a_n P_n(\mu_0)\right|}. \quad (44)$$

This expression for the Nusselt number is valid for all values of the contact angle $\theta_0 = \cos^{-1}\mu_0$ in the range $[0, \pi/2]$ and is presented as a function of θ_0 for values of the Biot number ranging from 0.05 to 20.0 in Fig. 4. As θ_0 approaches 0, the Nusselt number approaches the exact value πBi for a flat-disk droplet.

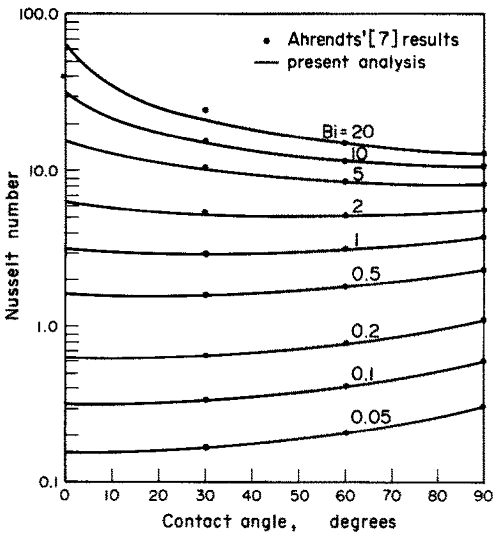

FIG. 4. Nusselt number as a function of contact angle for different Biot numbers.

Also, a comparison of these results with those obtained by Ahrendts [7] shows good agreement. It is interesting to see that for $Bi \sim 1$, the Nusselt number is almost invariant with the contact angle and, hence, experiments which are independent of this parameter are possible.

5. ERROR BOUNDS FOR NUSSELT NUMBER

An expression for the maximum error can be obtained from equation (41). The difference between the upper and lower bounds for Nu is given by

$$\Delta Nu = \frac{f(\mu_0)}{1-\bar{T}_h} - \frac{f(\mu_0)}{1-\underline{T}_h} \quad (45)$$

from which the maximum error becomes

$$\varepsilon = \frac{\Delta Nu}{\min(Nu)} = \frac{\bar{T}_h - \underline{T}_h}{1-\bar{T}_h}. \quad (46)$$

This bound on the error tends to zero, as θ_0 approaches $\pi/2$ because in this case $\bar{T}_h = \underline{T}_h = \frac{1}{2}$, i.e. for $\theta_0 = \pi/2$ the expression (44) for Nu is exact and for other contact angles close to $\pi/2$, ε is a good estimate of the error. However, for small contact angles ε becomes quite large but the actual error is quite small since Nu given by (44) also approaches the exact value for $\theta_0 = 0$.

6. ESTIMATE OF DROPLET GROWTH RATE

Under the quasisteady approximation used here, it can be assumed that the heat conducted through the droplet is equal to the heat transferred to it by condensation. Therefore, if λ is the latent heat of vaporization, then

$$Q = \lambda\rho\frac{\mathrm{d}V}{\mathrm{d}t}, \quad (47)$$

where ρ is the density of the droplet liquid and V is the volume of the droplet.

Now in terms of the base radius R and the contact angle parameter μ_0

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{1}{3}\pi R^3\frac{(1-\mu_0)^2(2+\mu_0)}{(1-\mu_0^2)^{3/2}}\right|,$$

i.e.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \pi R^2\frac{(1-\mu_0)^2(2+\mu_0)}{(1-\mu_0^2)^{3/2}}\frac{\mathrm{d}R}{\mathrm{d}t}. \quad (48)$$

By making use of (40), (44), (47) and (48) it is not difficult to show that the droplet base radius increases at the rate

$$\frac{\mathrm{d}R}{\mathrm{d}t} \simeq \Delta T\frac{2h}{\lambda\rho}\frac{(1+\mu_0)^{1/2}}{(1-\mu_0)^{1/2}(2+\mu_0)} \times \frac{\left|(1-a_0)-(1+\mu_0)\sum_{n=1}^{\infty}\frac{a_n P_n'(\mu_0)}{n(n+1)}\right|}{\left|(1-a_0)-\sum_{n=1}^{\infty} a_n P_n(\mu_0)\right|}. \quad (49)$$

7. DISCUSSION

The mathematical model presented in this study is limited to cases in which the heat-transfer coefficient at the free surface (i.e. at the liquid–vapour interface) is finite. The case of infinite heat-transfer coefficient, i.e. zero interfacial resistance, may be physically possible and in such a situation, because the model predicts zero overall resistance across the droplet, further improvement is necessary. The difficulty arises in assuming a uniform base temperature which is a valid approximation if the resistance between the base and the coolant is negligible compared to the liquid–vapour interfacial resistance or if the condenser thickness is of the same or higher order as the droplet size [6]. For the physically impossible case in which the resistances at both the interfaces are zero, the edge temperature is discontinuous and the overall droplet resistance is also zero. Fatica and Katz [2] were able to find non-zero resistance in their analysis because they divided the droplet into a finite number of elements and added the resistances of the elements. However, it can be shown that in the limit of the number of elements going to infinity the calculated resistance must go to zero. Sugawara and Michiyoshi [4] and Nijaguna [3] also obtained nonzero resistance and in their cases the error was due to the premature truncation of the divergent series representing total flow (see Appendix B).

In the analysis by Umur and Griffith [5] a relation between the droplet radius of curvature and time was given for an arbitrary contact angle. This relation implies a nonzero total resistance in the limit of the liquid–vapor interfacial resistance going to zero. This nonzero resistance is quite meaningless from a mathematical point of view, unless it contains some parameters related to the condenser–surface resistance which it does not.

The good agreement between the results of the present analysis and the numerical solutions of Ahrendts [7] provides mutual confirmation. The analytic solutions have the usual advantages over the numerical results but more importantly they are also the groundwork for further contributions exploiting the use of differential inequalities for the analysis of droplets. Indeed, results similar to Appendix A are available for the usual parabolic operators which describe most diffusion-controlled mechanisms for heat and mass transfer, e.g. [10].

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APPENDIX A

Theorem (Adapted from [8])

Let E be an elliptic differential operator defined in the domain D and let B be a boundary operator defined on the boundary ∂D by

$$B[\omega] = \begin{cases} \left(\dfrac{\partial \omega}{\partial n} + a\omega\right)\Big|_{\partial D_1}, \\ \omega|_{\partial D_2}, \end{cases} \tag{A1}$$

where $\partial D = \partial D_1 \cup \partial D_2$, ω is a continuous function in D, a is a non-negative function on ∂D, and $\partial/\partial n$ is the outward normal derivative.

Let u and v be continuous functions in D such that $E[u]$, $E[v]$, $B[u]$ and $B[v]$ all exist.

If

$$-E[u] \leqslant -E[v] \text{ and } B[u] \leqslant B[v], \tag{A2}$$

then

$$u \leqslant v. \tag{A3}$$

APPENDIX B

Unboundedness of Total Heat Flow for Discontinuous Boundary Temperature

For a hemispherical droplet with zero base temperature and unit temperature on the free surface, the total heat flow, Q through the droplet is given by [3, 4]

$$\frac{Q}{2\pi Rk} = \sum_{n=0}^{\infty} \frac{(4n+3)[(2n)!]^2(2n+1)}{(2n+2)^2 2^{4n}(n!)^4}, \tag{B1}$$

where k is the thermal conductivity of the droplet liquid, and R is the base radius.

It is not difficult to see that

$$\begin{aligned} \frac{(4n+3)[(2n)!]^2(2n+1)}{(2n+1)^2 2^{4n}(n!)^4} &\geqslant \frac{(3n+3)(n+1)[(2n)!]^2}{(2n+2)^2 2^{4n}(n!)^4} \\ &= \frac{3}{4}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \\ &= \frac{3}{4}\frac{(\frac{1}{2})_n(\frac{1}{2})_n}{(1)_n n!}. \end{aligned} \tag{B2}$$

However, the complete elliptic integral of the first kind is given by

$$K(z) = \int_0^{\pi/2} \frac{d\phi}{(1-z^2\sin^2\phi)^{1/2}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(\frac{1}{2})_n}{(1)_n n!}z^{2n}. \tag{B3}$$

Therefore, by letting $z = 1$ in (B3) and using (B1) and (B2) one can show that

$$\frac{Q}{kR} \geqslant 3\int_0^{\pi/2}\frac{d\phi}{\cos\phi} = -3\log\left|\tan\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\right|\Bigg|_0^{\pi/2} \to \infty$$

Therefore,

$$Q \to \infty \quad \text{for} \quad kR > 0. \tag{B4}$$

TRANSFERT THERMIQUE PAR CONDENSATION EN GOUTTES EN UTILISANT DES INEGALITES DIFFERENTIELLES

Résumé—On calcule analytiquement le nombre de Nusselt quasi statique pour une goutte de condensat, avec un angle de contact arbitraire, dans le domaine $[0, \pi/2]$ et en utilisant un segment sphérique. On suppose une température uniforme à la base et, pour assurer la conservation du flux thermique total, on utilise la condition de convection à la surface libre. On utilise des inégalités différentielles pour obtenir des limites d'erreur qui tendent vers zéro quand l'angle de contact tend vers $\pi/2$. Pour des petits angles de contact le nombre de Nusselt calculé tend vers la valeur exacte correspondant à une goutte en forme de disque plat.

DIE BERECHNUNG DES WÄRMEÜBERGANGS IN KONDENSATTROPFEN MIT HILFE VON DIFFERENTIAL-UNGLEICHUNGEN

Zusammenfassung—Unter Voraussetzung einer kugelförmigen Oberfläche wird die quasistationäre Nusselt–Zahl für einen Kondensattropfen mit beliebigem Randwinkel zwischen 0 und 2π analytisch berechnet. Die Fußtemperatur wird als gleichförmig angenommen; an der freien Oberfläche wird die Randbedingung dritter Art angesetzt. Mit Hilfe von Differential-Ungleichungen werden Fehlergrenzen abgesteckt; mit Annäherung des Randwinkels gegen $\pi/2$ gehen sie gegen Null. Für kleine Randwinkel nähert sich die berechnete Nusselt–Zahl dem exakten, für einen flachscheibigen Tropfen gültigen Wert.

РАСЧЕТ ХАРАКТЕРИСТИК ПЕРЕНОСА ТЕПЛА ЧЕРЕЗ КАПЕЛЬНЫЙ КОНДЕНСАТ С ПОМОЩЬЮ ДИФФЕРЕНЦИАЛЬНЫХ НЕРАВЕНСТВ

Аннотация — Используя сферическую геометрию сегмента, рассчитывается аналитически квазистационарное число Нуссельта для капли конденсата при произвольном значении краевого угла в диапазоне [0, $\pi/2$]. Делается допущение о постоянной температуре у основания капли, а для выполнения условия ограниченности общего теплового потока задано наличие конвекции на свободной поверхности. Дифференциальные неравенства используются для получения порога погрешностей, который приближается к нулю по мере того, как краевой угол стремится к $\pi/2$. Для небольших краевых углов расчетное число Нуссельта стремится к точному значению для капли в форме плоского диска.